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ON (2, n)-GROUPS RELATED TO FIBONACCI GROUPS

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ABSTRACT
In this paper, we study certain groups G generated by two elements a and b of orders 2 and n respectively subject to one further defining relation, and determine their structure. We also point out certain connections between these groups and the Fibonacci groups $F(r, n)$.

1. Introduction

In this paper, we shall be interested in groups defined by presentations of the form:

$$\langle a, b : a^2 = b^n = w = 1 \rangle,$$

where $n > 1$, w is a word in a and b, and the exponent sum of b in w is zero. The simplest such case would be:

$$\langle a, b : a^2 = b^n = ab^i ab^j = 1 \rangle$$

with $i + j = 0$, and such a group would be infinite if $(i, n) > 1$ or abelian of order $2n$ if $(i, n) = 1$. The next case would be:

$$\langle a, b : a^2 = b^n = ab^i ab^j ab^k = 1 \rangle$$

with $i + j + k = 0$. In this case, by replacing k by $k + n$, we may assume that $i + j + k = n$, and then our group is a homomorphic image of the group with presentation:

$$\langle a, b : a^2 = b^{2n} = ab^i ab^j ab^k = 1 \rangle,$$

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i.e. the group $H^{i,j,k}$, whose structure was determined in [2]. In this paper, we study the groups with presentations of the form:

$$\langle a, b : a^2 = b^n = ab^hab^iab^jab^k = 1 \rangle,$$

where $h + i + j + k = 0$, and determine all the possibilities with h, i, j, k in $\{1, -1, 2, -2\}$. We also point out some connections between these groups and the *Fibonacci groups* $F(r,n)$ with presentations:

$$\langle x_1, x_2, \ldots, x_n : x_1x_2\cdots x_r = x_{r+1},$$
$$x_2x_3\cdots x_{r+1} = x_{r+2}, \ldots, x_nx_1x_2\cdots x_{r-1} = x_r \rangle,$$

where the subscripts are taken to be reduced modulo n, and the semi-direct products $E(r, n)$ of $F(r, n)$ with the cyclic group of order n, which have presentations:

$$\langle b, c : bc^r = cb^r, b^n = 1 \rangle.$$

(See [3, 5, 6, 7] for further details.) The connection between groups of this form and the Fibonacci groups was also exploited in [9], where the groups with presentations:

$$\langle a, b : a^2 = b^n = abab^rab^{-r}ab^{-1} = 1 \rangle$$

were shown to be infinite if $(r + 1, n) > 3$ or if $(r + 1, n) = 3$ with n even, and it was deduced that the Fibonacci groups $F(r, n)$ are infinite in these cases.

The notation used in this paper is reasonably standard. We use $|.|$ to denote either the order of a group or the modulus of an integer, the context hopefully making it clear as to which was intended. If G and H are groups, we let $G \times H$ denote the direct product and $G * H$ the free product of G and H. For any group G, G' denotes the commutator subgroup of G, and, if x and y are elements of G, x^y denotes $y^{-1}xy$. We let C_n, D_n and E_n denote the cyclic, dihedral and elementary abelian groups respectively of order n, and A_n the alternating group of degree n. Lastly, (g_n) denotes the *Lucas sequence* of numbers defined inductively by:

$$g_1 = 1, \quad g_2 = 3,$$
$$g_n = g_{n-2} + g_{n-1} \quad (n > 2),$$

and, for any integers $a_1, a_2, \ldots, a_n$, we let $(a_1, a_2, \ldots, a_n)$ denote their highest common factor.

2. First reductions

In this section, we reduce the problem a little by pointing out certain isomorphisms between some of the groups under consideration. Let $G(n; h, i, j, k)$ denote the group with presentation:

$$\langle a, b : a^2 = b^n = ab^hab^iab^jab^k = 1 \rangle.$$

Then we first prove:

PROPOSITION 2.1. (i) *If n is even and $|h| = |i| = |j| = |k| = 2$, then $G(n; h, i, j, k)$ is infinite.*

(ii) *If n is odd and $|h| = |i| = |j| = |k| = 2$, then $G(n; h, i, j, k)$ is isomorphic to $G(n; h/2, i/2, j/2, k/2)$.*

PROOF. (i) follows easily from adding the relation $b^2 = 1$ to:

$$\langle a, b : a^2 = b^n = ab^hab^iab^jab^k = 1 \rangle$$

to get:

$$\langle a, b : a^2 = b^2 = 1 \rangle,$$

which is a presentation for the infinite group $C_2 * C_2$. For (ii), if n is odd, replace b by $c = b^2$ to yield the presentation:

$$\langle a, c : a^2 = c^n = ac^{h/2}ac^{i/2}ac^{j/2}ac^{k/2} = 1 \rangle,$$

and hence the result.

PROPOSITION 2.2. *$G(n; h, i, j, k)$ is isomorphic to $G(n; i, j, k, h)$.*

The proof of this is immediate.

In view of Propositions 2.1 and 2.2, we may assume that $h = 1$ if we wish to determine which of the $G(n; h, i, j, k)$ under consideration are finite. So let $G(n; i, j, k)$ denote the group defined by the presentation:

$$\langle a, b : a^2 = b^n = abab^iab^jab^k = 1 \rangle.$$

We now have:

PROPOSITION 2.3. *$G(n; i, j, k)$ is isomorphic to $G(n; k, j, i)$.*

PROOF. The relation:

$$abab^iab^jab^k = 1$$

is equivalent to:

$$ab^{-1}ab^{-k}ab^{-j}ab^{-i} = 1.$$

Replacing b by b^{-1} yields the result.

In view of Proposition 2.3, we now have five possible cases to consider:

$$G(n: 1, -1, -1) \simeq G(n; -1, -1, 1),$$

$$G(n; 2, -1, -2) \simeq G(n; -2, -1, 2),$$

$$G(n; -1, 1, -1),$$

$$G(n; -2, 2, -1) \simeq G(n; -1, 2, -2),$$

$$G(n; 2, -2, -1) \simeq G(n; -1, -2, 2).$$

We shall deal with the first two cases in Section 3, and the remaining three cases in Sections 4 to 6 respectively.

3. The groups $G(n; 1, -1, -1)$ and $G(n; 2, -1, -2)$

To investigate the structure of the groups $G(n; 1, -1, -1)$ and $G(n; 2, -1, -2)$, we first prove:

PROPOSITION 3.1. *If $(i, n) = (j, n) = 1$, then the groups $G(n; i, -1, -i)$ and $G(n; j, -1, -j)$ are isomorphic and metabelian of order $2n^2$.*

PROOF. Let us consider the group $G = G(n; i, -1, -i)$ with presentation:

$$\langle a, b : a^2 = b^n = abab^iab^{-1}ab^{-i} = 1 \rangle.$$

Let

$$a_1 = bab^{-1}, \quad a_2 = b^2ab^{-2}, \quad \ldots, \quad a_{n-1} = b^{n-1}ab^{1-n},$$

$N = \langle a, a_1, a_2, \ldots, a_{n-1} \rangle$. Then N is normal in G of index n, and may be easily checked to have presentation:

$$\langle a, a_1, a_2, \ldots, a_{n-1} : a^2 = a_1^2 = a_2^2 = \cdots = a_{n-1}^2$$

$$= aa_1a_{i+1}a_i = a_1a_2a_{i+2}a_{i+1} = \cdots = a_{n-1}aa_ia_{i-1} = 1 \rangle,$$

where subscripts are reduced modulo n. The last n relations may be written in the form:

$$aa_1 = a_ia_{i+1}, a_1a_2 = a_{i+1}a_{i+2}, \ldots, a_{n-1}a = a_{i-1}a_i,$$

and, if $(i, n) = 1$, we have:

$$aa_1 = a_i a_{i+1} = a_{2i} a_{2i+1} = \cdots,$$

in other words:

$$(*) \qquad aa_1 = a_1 a_2 = a_2 a_3 = \cdots = a_{n-2} a_{n-1} = a_{n-1} a.$$

The action of b on N is determined by:

$$a \to a_1$$

$$a_1 \to a_2 = a_1 a a_1,$$

which is independent of i, thus showing that the structure of G is independent of i, and hence all such groups are isomorphic. The relations $(*)$ give:

$$a_2 = a_1 a a_1,$$

$$a_3 = a_2(aa_1) = a_1(aa_1)^2,$$

$$a_4 = a_3(aa_1) = a_1(aa_1)^3,$$

$$\vdots$$

$$a_{n-1} = a_{n-2}(aa_1) = a_1(aa_1)^{n-2},$$

and then:

$$a = a_{n-1} a a_1 = a_1(aa_1)^{n-1},$$

in other words:

$$(aa_1)^n = 1.$$

So N has presentation:

$$\langle a, a_1 : a^2 = a_1^2 = (aa_1)^n = 1 \rangle,$$

and is thus isomorphic to D_{2n}. Hence $|G| = 2n^2$, and, since $[G, G'] = 2n$, G' is isomorphic to C_n, and thus G is metabelian.

From Proposition 3.1, we may now deduce:

Theorem 3.2. (i) $G(n; 1, -1, -1)$ *is metabelian of order* $2n^2$.
(ii) $G(n; 2, -1, -2)$ *is isomorphic to* $G(n; 1, -1, -1)$ *for n odd*.
(iii) $G(n; 2, -1, -2)$ *is infinite for n even*.

Proof. Parts (i) and (ii) follow immediately from Proposition 3.1. As for (iii), if n is even, then we may add the relation $b^2 = 1$ to the relations for $G(n; 2, -1, -2)$ to get a homomorphic image with presentation:

$$\langle a, b : a^2 = b^2 = 1 \rangle,$$

and hence the group is infinite.

4. The groups $G(n; -1, 1, -1)$

These groups are, in general, infinite, as is shown by:

THEOREM 4.1. (i) *$G(2; -1, 1, -1)$ is isomorphic to D_8.*
(ii) *$G(3; -1, 1, -1)$ is isomorphic to $A_4 \times C_2$.*
(iii) *$G(n; -1, 1, -1)$ is infinite for $n > 3$.*

PROOF. The group $G(n; -1, 1, -1)$ has presentation:

$$\langle a, b : a^2 = b^n = abab^{-1}abab^{-1} = 1 \rangle.$$

As in the proof of Proposition 3.1, let $a_1 = bab^{-1}, a_2 = b^2ab^{-2}, \ldots, a_{n-1} = b^{n-1}ab^{1-n}$, $N = \langle a, a_1, \ldots, a_{n-1} \rangle$. Then N has presentation:

$$\langle a, a_1, a_2, \ldots, a_{n-1} : a^2 = a_1^2 = a_2^2 = \cdots = a_{n-1}^2$$
$$= (aa_1)^2 = (a_1a_2)^2 = \cdots = (a_{n-2}a_{n-1})^2 = (a_{n-1}a)^2 = 1 \rangle.$$

If $n = 2$, then N has presentation:

$$\langle a, a_1, : a^2 = a_1^2 = (aa_1)^2 = 1 \rangle,$$

so that N is isomorphic to E_4, and, since conjugation by b interchanges a and a_1, $G(2; -1, 1, -1)$ is isomorphic to D_8. If $n = 3$, then N has presentation:

$$\langle a, a_1, a_2 : a^2 = a_1^2 = a_2^2 = (aa_1)^2 = (a_1a_2)^2 = (a_2a)^2 = 1 \rangle,$$

so that N is isomorphic to E_8. Let:

$$c_1 = aa_1, \quad c_2 = a_1a_2, \quad c_3 = aa_1a_2.$$

Then $bc_1b^{-1} = c_2$, $bc_2b^{-1} = c_1c_2$, and $bc_3b^{-1} = c_3$, so that

$$G(3; -1, 1, -1) = \langle c_1, c_2, b \rangle \times \langle c_3 \rangle$$

is isomorphic to $A_4 \times C_2$. If $n > 3$, we argue as in [9]; add the relations:

$$a_1 = a_3 = a_4 = a_5 = \cdots = a_{n-1} = 1$$

to those for N to get a homomorphic image with presentation:

$$\langle a, a_2 : a^2 = a_2^2 = 1\rangle,$$

and hence the group is infinite.

5. The groups $G(n;\ -2, 2,\ -1)$

The groups $G(n;\ -2, 2,\ -1)$ give us our second infinite class of finite groups:

THEOREM 5.1. *$G(n;\ -2, 2,\ -1)$ is metabelian and has order* $2n(2^n - (-1)^n)/3$.

PROOF. Let $G = G(n;\ -2, 2,\ -1)$ with presentation:

$$\langle a, b : a^2 = b^n = abab^{-2}ab^2ab^{-1} = 1\rangle.$$

We introduce a generator $x = ab^{-1}ab$:

$$\langle a, b, x : a^2 = b^n = 1, x = ab^{-1}ab, abab^{-2}ab^2ab^{-1} = 1\rangle.$$

Using $a^2 = 1$, we may rewrite the last relation as:

$$b^{-2}ab^2ab^{-1}aba = 1,$$

i.e.

$$b^{-1}ab^2 = bab^{-1}aba,$$

i.e.

$$b^{-1}ab^{-1}ab^2 = (b^{-1}aba)^2,$$

i.e.

$$b^{-1}xb = x^{-2},$$

so that G has presentation:

$$\langle a, b, x : a^2 = b^n = 1, x = ab^{-1}ab, b^{-1}xb = x^{-2}\rangle.$$

Let $N = \langle b, x\rangle$. Then $a^{-1}xa = x^{-1}$, $a^{-1}ba = bx^{-1}$, so that N is normal in G and $[G : N] = 2$. Working as in [5, Chapter 7], we derive a presentation for N via the coset representatives $\{1, a\}$. N is generated by elements of the form $uv(\overline{uv})^{-1}$, where $u \in \{1, a\}$, $v \in \{a, b, x\}$, and $\overline{uv}$ is the element of $\{1, a\}$ such that $uv \in N\overline{uv}$. The relations for N are then all expressions of the form $uru^{-1} = 1$, where $u \in \{1, a\}$ and $r \in \{a^2, b^n, xb^{-1}aba^{-1}, b^{-1}xbx^2\}$. This gives generators $t = a^2$, b, $c = aba^{-1}$, x, $y = axa^{-1}$, and relations:

$$t = b^n = c^n = xb^{-1}c = yc^{-1}tbt^{-1} = b^{-1}xbx^2 = c^{-1}ycy^2 = 1.$$

Eliminating the trivial generator t, we have as presentation for N:

$$\langle b, c, x, y : b^n = c^n = 1, c = bx^{-1}, y = b^{-1}c, b^{-1}xbx^2 = c^{-1}ycy^2 = 1\rangle.$$

We eliminate $c = bx^{-1}$ to get:

$$\langle b, x, y : b^n = (bx^{-1})^n = 1, y = x^{-1}, b^{-1}xbx^2 = xb^{-1}ybx^{-1}y^2 = 1\rangle.$$

We now eliminate $y = x^{-1}$ to get:

$$\langle b, x : b^n = (bx^{-1})^n = b^{-1}xbx^2 = xb^{-1}x^{-1}bx^{-3} = 1\rangle.$$

The last relation is redundant via the penultimate, and we have:

$$\langle b, x : b^n = (bx^{-1})^n = 1, b^{-1}xb = x^{-2}\rangle.$$

The relation $(bx^{-1})^n$ may be rewritten as $(xb^{-1})^n$, and then as:

$$x \,.\, b^{-1}xb \,.\, b^{-2}xb^2 \,.\, \cdots \,.\, b^{-(n-1)}xb^{n-1} = 1,$$

which, in view of the last relation, is equivalent to:

$$x \,.\, x^{-2} \,.\, x^4 \,.\, x^{-8} \,.\, \cdots \,.\, x^{(-2)^{n-1}} = 1,$$

i.e. $x^k = 1$, where:

$$k = |1 - 2 + 4 - 8 + \cdots + (-2)^{n-1}|$$
$$= (2^n - (-1)^n)/3.$$

So N has presentation:

$$\langle b, x : b^n = x^k = 1, b^{-1}xb = x^{-2}\rangle,$$

and N is metacyclic of order nk. Thus G is metabelian of order $2nk$ as required.

NOTE 5.2. The group $G = G(n; -2, 2, -1)$ has presentation:

$$\langle a, b : a^2 = b^n = 1, abab^{n-2} = bab^{n-2}a\rangle.$$

Let $c = aba$, $M = \langle b, c\rangle$. Then M is normal in G of index 2, and has presentation:

$$\langle b, c : b^n = c^n = 1, cb^{n-2} = bc^{n-2}\rangle.$$

By Theorem 5.1, M is metacyclic of order nk, where $k = (2^n - (-1)^n)/3$. As in Section 1, let F be the Fibonacci group $F(n-2, n)$, E be the group $E(n-2, n)$, so that $[E : F] = n$. Then E has presentation:

$$\langle b, c : b^n = 1, cb^{n-2} = bc^{n-2}\rangle,$$

so that M is a homomorphic image of E. Now $|F/F'| = (n-3)k$ (e.g. see [4, Corollary 4] or [5, Chapter 16, Theorem 7]), and hence $|E/F'| = (n-3)nk$.

So the normal closure of $\langle c^n \rangle$ in E has index nk in E and contains F' as a subgroup of index $n-3$.

6. The groups $G(n; 2, -2, -1)$

The groups $G(n; 2, -2, -1)$ with n odd give us our third class of finite groups, but are, in general, infinite when n is even. More precisely, we have:

THEOREM 6.1. (i) *$G(2; 2, -2, -1)$ is isomorphic to E_4.*
(ii) *$G(4; 2, -2, -1)$ is metabelian of order* 40.
(iii) *$G(n; 2, -2, -1)$ is infinite for n even, $n > 4$.*
(iv) *$G(n; 2, -2, -1)$ has order $2ng_n$ for n odd.*

Before proving Theorem 6.1, we make some observations, similar to those made in Note 5.2 and [7, Section 4]. The group $G(n; 2, -2, -1)$ has presentation:

$$\langle a, b : a^2 = b^n = 1, abab^2 = bab^2a \rangle.$$

As in 5.2, let $c = aba$, M be the normal subgroup $\langle b, c \rangle$ of index 2. Then M has presentation:

$$\langle b, c : b^n = c^n = 1, cb^2 = bc^2 \rangle,$$

which is a homomorphic image of the group $E = E(2, n)$ with presentation:

($) $$\langle b, c : b^n = 1, cb^2 = bc^2 \rangle.$$

Now, if $F = F(2, n)$, then F/F' has order g_n for n odd (see [3] or [5, Chapter 16, Exercise 4]), and so E/F' has order ng_n. Thus Theorem 6.1 (iv) gives that the normal closure of $\langle c^n \rangle$ in E is F' for n odd. Conversely, 6.1 (iv) may be derived from this fact. For if we let F have presentation:

$$\langle x_1, x_2, \ldots, x_n : x_1x_2 = x_3, x_2x_3 = x_4, \ldots, x_nx_1 = x_2 \rangle,$$

and let b denote the automorphism of F defined by:

$$x_1 \to x_2, \quad x_2 \to x_3, \ldots, x_{n-1} \to x_n, \quad x_n \to x_1,$$

c denote bx_1^{-1}, then $E = E(2, n)$ has presentation ($) (see [5, Chapter 16] for example). The relation $c^n = 1$ then corresponds to $(bx_1^{-1})^n = 1$, i.e. to:

$$bx_1^{-1}b^{-1} . b^2x_1^{-1}b^{-2} . \cdots . b^{n-1}x_1^{-1}b^{1-n} . b^n . x_1^{-1} = 1,$$

i.e., given that $b^n = 1$, to:

$$x_n^{-1}x_{n-1}^{-1}\cdots x_2^{-1}x_1^{-1}=1,$$

i.e.

$$x_1x_2\cdots x_{n-1}x_n=1.$$

Now, since:

$$x_1x_2=x_3,\quad x_2x_3=x_4,\ldots,x_nx_1=x_2,$$

abelianizing and multiplying these relations together yields:

$$(\bar{x}_1\bar{x}_2\cdots\bar{x}_n)^2=\bar{x}_1\bar{x}_2\cdots\bar{x}_n,$$

i.e.

$$\bar{x}_1\bar{x}_2\cdots\bar{x}_n=1$$

in $\bar{F}=F/F'$. So $x_1x_2\cdots x_n\in F'$, and hence the normal closure D of $\langle c^n\rangle$ in E is contained in F'. If D were equal to F', then G would be an extension of E/F' by C_2, and Theorem 6.1 (iv) would follow.

We should point out that the situation is quite different for n even. Here the relations:

$$b^n=1,\qquad cb^2=bc^2,$$

give:

$$\begin{aligned}(c^{-2}b)^{-1}b^2(c^{-2}b)&=b^{-1}c(cb^2c^{-2})b\\&=b^{-1}cb^2\\&=c^2,\end{aligned}$$

so that $c^n=(c^2)^{n/2}=(c^{-2}b)^{-1}(b^2)^{n/2}(c^{-2}b)=1$. So, for n even, $G(n;2,-2,-1)$ contains $E(2,n)$ as a subgroup of index 2, and has derived subgroup isomorphic to $F(2,n)$ with index $2n$. Since $F(2,2)$ is trivial and $F(2,4)$ isomorphic to C_5 (see [3], [5, Chapter 16] or [7]), Theorem 6.1 (i) and (ii) now follow. Since $F(2,n)$ is infinite for n even, $n>4$ (see [1, 3, 8]), Theorem 6.1 (iii) also follows. It remains to prove (iv).

So let us now consider the group $G=G(n;2,-2,-1)$ with presentation:

$$\langle a,b: a^2=b^n=abab^2ab^{-2}ab^{-1}=1\rangle$$

with n odd. Let $x=bab^{-1}a$, $y=ab^{-1}ab$, $n=2k-1$. Then the last relation gives:

$$ba\,.\,b^2\,.\,ab^{-1}=(ab)^2,$$

and so $(ab)^{2k}=ba\,.\,b^{2k}\,.\,ab^{-1}=babab^{-1}$, so that:

$$(ab)^n=(ab)^{2k}(ab)^{-1}=babab^{-2}a.$$

Now $(ab)^{2n} = ba \, . \, b^{2n} \, . \, ab^{-1} = 1$, so that $(babab^{-2}a)^2 = 1$, i.e. $(bab^{-1}a \, . \, ab^{-1}ab \, . \, a)^2 = 1$, i.e. $(xya)^2 = 1$. Since:

$$axa = abab^{-1} = x^{-1},$$

$$aya = b^{-1}aba = y^{-1},$$

we have that $xyx^{-1}y^{-1} = 1$, i.e. $xy = yx$.

Now the relation $abab^2ab^{-2}ab^{-1} = 1$ gives us that $y = b^2ab^{-2}a$. So $b^{-1}xb = y$, $b^{-1}yb = bab^{-2}ab = bab^{-1}a \, . \, ab^{-1}ab = xy$, and hence $N = \langle x, y \rangle$ is normal in G. Since $x^b = y$, $y^b = xy$ and $b^n = 1$, N is a homomorphic image of F. Since N is abelian, N is a homomorphic image of F/F'. But we already know that G is an extension of F/D by C_{2n}, where D is contained in F'. Hence $D = F'$, and the result follows.

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